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Abstract. This paper studies divergence in solutions of p-adic linear local differential equations. Such divergence is related to the notion of p-adic Liouville numbers. Also, the influence of the divergence on the differential Galois groups of such differential equations is explored. A complete result is given for second order equations and a conjecture for higher order equations is proposed.

Keywords: p-Adic analysis, differential equations, Liouville numbers, divergence.

1. INTRODUCTION

Let k be a complete non-archimedean valued field containing the field of p-adic numbers. We suppose that k is algebraically closed. Let $k\{x\}$ denote the ring of convergent power series over k and let $K = k(\{x\})$ be its field of fractions. The aim of this paper is to investigate the divergence of solutions of linear differential equations and to compute differential Galois groups. This project is related in spirit to the work of J.P. Ramis and J. Martinet which is concerned with the same questions but then with base field the complex numbers (see [2], [5] and [4]). The main observation is that in the complex case, the divergence of solutions is due to the singularity of the equation and that in particular regular singular equations do not produce divergence. In contrast to this, in the p-adic case it is not the irregularity that produces divergence. In fact, the divergence in the p-adic situation comes from the regular singular case when p-adic Liouville numbers are present. This paper can be seen as a continuation of [1] and [3]. The analysis of the divergence for p-adic differential equations and the contribution of this to the differential Galois group is much more involved than in the complex case. There are two reasons for this, namely the divergence cannot be classified in levels and moreover there is at present no asymptotic theory on sectors (or the like) available in the p-adic case.

2. NOTATION

- $\mathbb{Z}_p$, $\mathbb{Q}_p$: the ring of p-adic integers and its quotient field
- k: a complete and algebraically closed field containing $\mathbb{Q}_p$
- $\widehat{K} = k((x))$, $K = k(\{x\})$: the field of formal Laurent series and the subfield of convergent Laurent series

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- $k[[x]] \subset \widehat{K}$, $k\{x\} \subset K$: the subrings of functions without pole at $x = 0$
- δ: the derivation $x\frac{d}{dx}$ on the above rings and fields

3. p-ADIC LIOUVILLE NUMBERS

Whereas the classical Liouville numbers are real numbers that can be rapidly approximated by rational numbers, the p-adic Liouville numbers are those numbers that can be rapidly approximated by *positive integers* for the p-adic valuation. The precise definition is the following. An element $\lambda \in k$ is called a *p-adic Liouville number* if $\liminf_{n\to+\infty} |\lambda - n|^{1/n} = 0$. The terminology is introduced in [C] and was rediscovered in [P80]. In the latter the above definition is given because it is better adapted to differential equations. Indeed, consider the inhomogeneous differential equation

$$(\delta - \lambda)y = \frac{1}{1-x}, \text{ with } \lambda \notin \mathbb{Z}. \tag{1}$$

This equation has a unique formal solution, namely $\sum_{n\geq 0} \frac{1}{n-\lambda} x^n$. This solution is divergent precisely when λ is a p-adic Liouville number.

It seems that the set $\mathscr{L}$ of p-adic Liouville numbers does not have much structure. One can make the following observations

1. $\mathscr{L} \subset \mathbb{Z}_p$
2. $\mathscr{L}$ has measure 0 for the (real) Haar measure on $\mathbb{Z}_p$
3. If $a \in \mathscr{L}$ and $n, m \in \mathbb{Z}$ with $m > 0$ then $n + ma \in \mathscr{L}$
4. $\mathscr{L} \neq -\mathscr{L}$ and $\mathscr{L} \cap -\mathscr{L} \neq \emptyset$
5. Every $a \in \mathscr{L}$ is transcendent over $\mathbf{Q}$

Let a subset $\{\lambda_1, \dots, \lambda_s\}$ of k be given such that $\lambda_i - \lambda_j \in \mathbb{Z}$ implies that $i = j$ (this condition will correspond to a differential equation being non-resonant). We associate to this set an oriented graph E. The vertices of E are $\{v_1, \dots, v_s\}$ and (v_i, v_j) (with $i \neq j$) is an oriented edge if and only if $\lambda_j - \lambda_i \in \mathscr{L}$.

Theorem 1. *Let E be a finite oriented graph such that:*

(a) *The two ends of every oriented edge are distinct.*

(b) *For every pair of distinct vertices a, b there is at most one oriented edge from a to b.*

There exists a finite subset $\{\lambda_1, \dots, \lambda_s\}$ of k such that $\lambda_i - \lambda_j \in \mathbb{Z}$ implies that $i = j$ and such that E is its associated oriented graph.

The proof is rather intricate and combinatorial.

4. DIFFERENTIAL EQUATIONS OVER p-ADIC FIELDS

A differential operator

$$L = \delta^n + a_{n-1}\delta^{n-1} + \ldots + a_0$$

with coefficients in K (or in $\widehat{K}$) is called *regular singular* if and only if all a_i are in $k\{x\}$ (or in $k[[x]]$), that is, have no pole at $x = 0$. L is called *irregular* when it is not regular singular.

Irreducible irregular operators with coefficients in K have the property that the formal solutions of the equation $L(y) = f$ with $f \in K$ are automatically convergent. One can, for example, easily verify this for the equation

$$(\delta - x^{-1})y = \sum_{n\geq 0} b_n x^n \tag{2}$$

This surprising result is completely opposite to the complex case. In fact, in questions on divergence, the regular singular equations over K will be the most interesting, as already can be seen in the simple example (1). Also this is very different from the complex case, where divergence in regular singular equations with convergent coefficients does not occur.

More precisely, using some p-adic functional analysis, one can show:

Proposition 2. *Let $f \in \widehat{K}$ be a divergent solution of a linear differential equation with coefficients in K. Then the minimal homogeneous equation satisfied by f has the form*

$$(\delta^m + a_{m-1}\delta^{m-1} + \cdots + a_0)y = 0 \textit{ with all } a_i \in K$$

with all $a_i \in k\{x\}$ and such that the polynomial

$$P(T) := T^m + a_{m-1}(0)T^{m-1} + \cdots + a_0(0) \in k[T]$$

has a root in **Z** *and has a root which is a p-adic Liouville number.*

Hence, from now on we will mainly consider regular singular differential equations. Using the definition of a p-adic Liouville number one can verify the following:

Proposition 3. *Consider the operator $L = \delta^m + a_{m-1}\delta^{m-1} + \cdots + a_0$ with coefficients in $k\{x\}$ and the polynomial*

$$P(T) = T^m + a_{m-1}(0)T^{m-1} + \cdots + a_0(0) \in k[T].$$

If no zero of P is a p-adic Liouville number, then all formal solutions of the inhomogeneous equations $L(y) = f$ with $f \in K$ are convergent.

Example 4. We note that the solution $y = \sum_{n\geq 0} \frac{1}{n-\lambda} x^n$ of the inhomogeneous equation $(\delta - \lambda)y = \frac{1}{1-x}$ is also a solution of the monic homogeneous equation $(\delta^2 + \frac{-\lambda + x(1-\lambda)}{1-x}\delta + \frac{\lambda x}{1-x})y = 0$. The corresponding polynomial $P(T) = T^2 - \lambda T$ has zeros $\lambda, 0$. Therefore the propositions predict that y is convergent if and only if $\lambda \in \mathscr{L}$, in accordance with the definition of $\mathscr{L}$.

5. REGULAR SINGULAR DIFFERENTIAL MODULES

To get a better understanding of divergence in regular singular equations, it is convenient to pass from the language of differential equations to the language of differential modules. These form a "coordinate-free" representation of differential equations and moreover, they permit the use of several constructions from linear algebra. For more details, see [4].

A differential module over K (or any differential field) is a finite dimensional vector space M over K together with an operation $\partial : M \to M$ satisfying the Leibnitz rule $\partial f = f\partial + \delta f$. A submodule $N \subset M$ is per definition a vector subspace of M that is preserved by ∂. The direct sum of two differential modules M_1 and M_2 is the direct sum of the underlying vector spaces on which ∂ acts as $\partial(m_1, m_2) = (\partial m_1, \partial m_2)$. It is also possible to define quotients, tensor products, symmetric products, internal Homs for differential modules (see [4]).

One can construct a scalar differential equation from the module if one can find a cyclic vector $e \in M$, that is, a vector e such that $e, \partial e, \partial^2 e, \ldots$ form a basis. Such a cyclic vector always exists. If the dimension of M is n, we then find that $\partial^n e$ can be written as $-a_0 e - a_1 \partial e - \cdots - a_{n-1}\partial^{n-1} e$, yielding the coefficients of a linear differential equation $L(y) = 0$ with

$$L = \delta^n + a_{n-1}\delta^{n-1} + \ldots + a_0.$$

From the above equation, one obtains a first order matrix equation by taking the vector $\mathbf{y} = (y, \delta y, \ldots, \delta^{n-1} y)$ to be the unknown. One obtains an equation of the form

$$\delta \mathbf{y} = A\mathbf{y},$$

with A an $n \times n$ matrix over K.

Such an equation in turn yields a differential module M by taking A to be the matrix of the action of ∂ on a basis of M.

Now, let M be a regular singular differential module, *i.e.*, a differential module associated with a regular singular equation. Then M has a ∂-invariant lattice $M^o = k\{x\}b_1 + \cdots + k\{x\}b_m$. With respect to this basis one can represent M by a matrix differential operator $\delta + A$ with $A \in \mathrm{Matr}(m, k\{x\})$. One can expand $A = A_0 + A_1 x + \cdots$ with all $A_i \in \mathrm{Matr}(m, k)$. The term A_0 is also the matrix of the operator ∂ acting upon M^o/xM^o. For a suitable lattice, the eigenvalues $\lambda_1, \ldots, \lambda_s$ of A_0 have the property that $\lambda_i - \lambda_j \notin \mathbb{Z}$ for $i \neq j$. The $\lambda_1, \ldots, \lambda_s$ are uniquely determined by M, up to translation over integers. For convenience, we will call them the *eigenvalues of M*.

In the complex case $\delta + A$ is equivalent to $\delta + A_0$ by a convergent transformation. Moreover, the conjugacy class of the monodromy matrix $e^{2\pi i A_0}$ classifies the regular singular differential module. In our situation this is not the case, due to "small denominators" and p-adic Liouville numbers.

One defines an oriented graph E attached to M as follows. The vertices of E are called $v_1, \ldots, v_s$. There is an oriented edge (v_i, v_j) with $i \neq j$ if and only if $\lambda_j - \lambda_i$ is a p-adic

Liouville number. Let $E_1, \ldots, E_r$ denote the connected components of the oriented graph E.

Proposition 5. *The regular singular differential module M has a unique direct sum decomposition $M = \oplus_{i=1}^{r} M_i$ such that M_i is a regular singular differential module with associated oriented graph E_i (and with the eigenvalues corresponding to this subset of E).*
In particular, suppose that the set of eigenvalues $\{\lambda_1, \ldots, \lambda_s\}$ of M has the property that $\lambda_i - \lambda_j \notin \mathscr{L}$ for all $i \neq j$. Then M has a basis over K such that the matrix of ∂ w.r.t. this basis is constant (i.e., has coefficients in k).

Proof. We sketch the proof. The formal decomposition (over $\widehat{K}$) always exists, moreover this decomposition is convergent if there are no p-adic Liouville numbers involved. Hence the decomposition according to the connected components E_i of E converges. □

The above proposition reduces the study of regular singular differential modules to the case where the associated oriented graph is connected. Our next aim is to produce submodules according to the structure of the connected oriented graph. The following proposition describes a collection of submodules of M in terms of the graph $E(M)$. For a *generic* M having a prescribed graph $E(M) = E$, the result is optimal, that is, the proposition describes *all* submodules.

Proposition 6. *Let the connected oriented graph E be associated to the regular singular differential module M. Suppose that the set of the vertices V of E is a disjoint union of two sets V_1, V_2 such that (v_2, v_1) is not an oriented edge for any $v_1 \in V_1,\ v_2 \in V_2$. Then there is a unique submodule $M_1 \subset M$ corresponding to the eigenvalues belonging to V_1.*

6. EXAMPLES OF DIFFERENTIAL GALOIS GROUPS

Consider a linear differential equation $L(y) = 0$ of order n and with coefficients in K. Then, of course, not all solutions of the equation need to exist in K, but over a suitably large differential field extension of K, the solution space becomes n-dimensional. In fact, there is a smallest such differential field extension, called the Picard-Vessiot field of the equation. The differential Galois group of L is defined to be the group of K-linear differential field automorphisms of the Picard-Vessiot field. It acts on the solution space V of the equation, and one knows that it is an algebraic subgroup of $\mathrm{GL}(V)$. We refer the reader to [4] for more information on differential Galois gruops.

In this section we compute the subgroups of SL_2 that are realizable as differential Galois group of a regular singular differential module over K.

Over $\widehat{K}$ - or equivalently, over $\mathbb{C}((x))$, since to the formal power series the topology of the field of constants does not matter - these have to be one of the following: finite cyclic, $\mathbf{G}_a$ (additive), $\mathbf{G}_m$ (multiplicative) or $\mathbf{G}_a \times \{\pm 1\}$.

Over K more subgroups are possible. Let M be a two-dimensional differential module over K. Take a δ-invariant lattice $M^o = k\{x\}e_1 + k\{x\}e_2$ and let λ_1, λ_2 denote the eigenvalues of δ operating on M^o/xM^o. Assume that $\lambda_2 - \lambda_1 \notin \mathbb{Z}$. The condition that the differential Galois group G is contained in $\mathrm{SL}_2(k)$ is easily seen to be equivalent to $\lambda_1 + \lambda_2 \in \mathbb{Z}$. There are several cases:

1. $\lambda_2 - \lambda_1 \notin \mathscr{L} \cup -\mathscr{L}$. So $E(M) = * \;\; *$
   In this case, by Proposition 5, M is a direct sum of one-dimensional modules and the differential Galois groups of M and of $\widehat{M}$ coincide. The list of possible subgroups is:
$$\mathbf{G}_m, \text{ finite cyclic }, \mathbf{G}_a \text{ and } \{\pm 1\} \times \mathbf{G}_a \;.$$
2. $\alpha := \lambda_2 - \lambda_1 \in \mathscr{L} \setminus -\mathscr{L}$ and M "generic". ($E(M) = * \to *$)
   Let $\lambda_1 + \lambda_2 = n \in \mathbb{Z}$. Then $\lambda_1 = -\alpha/2 + n/2$ and $\lambda_2 = \alpha/2 + n/2$. Since α is not rational one finds that the formal differential Galois group is $\mathbf{G}_m$. This is a subgroup of G. Since M is generic, it has only one non trivial submodule (by 6). One concludes that G is (conjugated to) the Borel subgroup $B \subset \mathrm{SL}_2(k)$.
3. $\lambda_2 - \lambda_1 \in \mathscr{L} \cap -\mathscr{L}$ and M "generic". ($E(M) = * \leftrightarrow *$)
   The formal differential Galois group is again $\mathbf{G}_m$. Since M is generic it has only trivial submodules (by 6). The differential Galois group G can only be SL_2 or the infinite dihedral group D_∞. In the latter case $N := K(x^{1/2}) \otimes M$, as differential module over $K(x^{1/2})$, has differential Galois group $\mathbf{G}_m$. Therefore N is a direct sum of two 1-dimensional subspaces. This is, however, excluded by Remark 7. We conclude that the differential Galois group of M is SL_2.

Remark 7. If the regular singular module M over $K = k(\{x\})$ has no submodules then the same holds for the module $M \otimes_K k(\{x^{1/n}\})$ over $k(\{x^{1/n}\})$, for any $n \geq 2$. The reason for this is that the eigenvalues get multiplied by n after this base change, and a positive integral multiple of a Liouville number is again a Liouville number (see §1).

Hence, we conclude that the list of realizable subgroups of SL_2 for regular singular equations is: $\mathrm{SL}_2,\ B,\ \mathbf{G}_m, \mathbf{G}_a,\ \{\pm 1\} \times \mathbf{G}_a,$ finite cyclic .

The subgroups of SL_2 that are realizable as differential Galois groups of irregular singular modules are $\mathbf{G}_m$ and the infinite dihedral group D_∞. It is interesting to compare this with the list of the realizable subgroups of SL_2 for equations over $\mathbb{C}(\{x\})$, which is

$$\mathrm{SL}_2,\ B,\ D_\infty,\ \mathbf{G}_m, \mathbf{G}_a,\ \{\pm 1\} \times \mathbf{G}_a,\ \text{ finite cyclic }.$$

In the complex case, the differential Galois group is generated, as algebraic group, by the formal differential Galois group and the Stokes matrices. The groups SL_2, B and D_∞ occur only for irregular singularities. SL_2 occurs when there are at least two distinct non-trivial Stokes matrices, and B occurs when there is only one non-trivial Stokes matrix. For the D_∞ group, there are no Stokes matrices. The group $\mathbf{G}_m$ occurs both in the irregular case with trivial Stokes matrices, or in the regular singular case. The last three groups only occur in the regular singular case.

One observes from the above that divergence due to p-adic Liouville numbers is the parallel of divergence caused by the complex Stokes matrices. The reader will note, however, that the former occurs for *regular* singular p-adic differential equations, while the latter occurs for *irregular* singular complex differential equations.

All this leads naturally to the following conjecture.

Conjecture 8. *Given a graph E as before, and a generic regular singular differential module M over K with $E(M) = E$. Let S be the solution space of M and $S = \oplus S_i$ the formal decomposition corresponding to the eigenvalues λ_i. The differential Galois group of M is the smallest algebraic group G such that*

- *G contains the formal differential Galois group,* i.e. *the differential Galois group of $M \otimes_K \widehat{K}$ over $\widehat{K}$,*
- *for every oriented edge (v_i, v_j) in E, the Lie algebra* $\mathrm{Lie}(G)$ *contains all linear maps*

$$S \overset{proj}{\to} S_j \overset{\alpha}{\to} S_i \subset S$$

  with $\alpha \in \mathrm{Hom}(S_j, S_i)$.

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